

NON-LOCAL CURVATURE AND TOPOLOGY OF LOCALLY CONFORMALLY FLAT MANIFOLDS

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ABSTRACT. In this paper, we focus on the geometry of compact conformally flat manifolds (M^n, g) with positive scalar curvature. Schoen-Yau proved that its universal cover $(\widetilde{M}^n, \tilde{g})$ is conformally embedded in \mathbb{S}^n such that M^n is a Kleinian manifold. Moreover, the limit set of the Kleinian group has Hausdorff dimension $< \frac{n-2}{2}$. If additionally we assume that the non-local curvature $Q_{2\gamma} > 0$ for some $1 < \gamma < 2$, then the Hausdorff dimension of the limit set is less than $\frac{n-2\gamma}{2}$. In fact, the above upper bound is sharp. As applications, we obtain some topological rigidity and classification theorems in lower dimensions.

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1. INTRODUCTION

Compact locally conformally flat manifolds with positive scalar curvature can be viewed as Kleinian manifolds by Schoen-Yau's fundamental work in [SY]. That is, if (M^n, g) is a compact locally conformally flat manifold with $R_g > 0$, then the universal cover $(\widetilde{M}^n, \tilde{g})$ can be conformally embedded in the standard sphere (\mathbb{S}^n, g_1) . Moreover, $\pi_1(M^n)$ is isomorphic to a Kleinian

group $\Gamma \leq \text{Conf}(\mathbb{S}^n)$ such that

$$\widetilde{M}^n \cong \Omega(\Gamma) \equiv \mathbb{S}^n \setminus \Lambda, \quad (1.1)$$

where $\Lambda \equiv \Lambda(\Gamma)$ is the limit set of the Kleinian group Γ . In [SY], Schoen-Yau also proved the following Hausdorff dimension estimate on the limit set Λ in the above setting,

$$\dim_{\mathcal{H}}(\Lambda) < \frac{n-2}{2}. \quad (1.2)$$

The above Hausdorff dimension estimate immediately gives homotopy vanishing and homology vanishing results, which are interesting topological obstructions for conformally flat manifolds with nonnegative scalar curvature (see [SY] for more details).

In this paper, we are trying to generalize the above theory to the fractional setting. In conformal geometry, scalar curvature R_g arises as the zeroth order term of the conformal Laplacian operator. More precisely, denote $J_g \equiv \frac{R_g}{2(n-1)}$, then

$$P_2 \equiv -\Delta_g + \frac{n-2}{2}J_g. \quad (1.3)$$

It is standard that the second order conformal Laplacian operator P_2 satisfies the following conformal covariance property: For $n \geq 3$, let $\hat{g} = v^{\frac{4}{n-2}}g$ and let \hat{P}_2 be the conformal Laplacian with respect to the conformal metric \hat{g} , then

$$\hat{P}_2(u) = v^{-\frac{n+2}{n-2}}P_2(uv). \quad (1.4)$$

The fourth order analogy of P_2 is called *Paneitz operator*, which is defined by

$$P_4(u) \equiv (-\Delta_g)^2 u + \text{Div}_g(4A_g \langle \nabla_g u, e_j \rangle e_j - (n-2)J_g \nabla_g u) + \frac{n-4}{2}Q_4 \cdot u, \quad (1.5)$$

where

$$A_g \equiv \frac{1}{n-2} \left(\text{Ric}_g - J_g \cdot g \right). \quad (1.6)$$

The above $Q_4 = (\frac{n-4}{2})^{-1}P_4(1)$ is called Branson's Q curvature. Similar to (1.4), Paneitz operator has the following conformal covariance property: For $n \geq 5$ and $\hat{g} = v^{\frac{4}{n-4}}g$, then

$$\hat{P}_4(u) = v^{-\frac{n+4}{n-4}}P_4(uv). \quad (1.7)$$

Chang-Hang-Yang studied the covering geometry similar to what Schoen-Yau did (see more details in [CHY]). In our paper, we focus on fractional order conformally covariant operator $P_{2\gamma}$ and the corresponding $Q_{2\gamma}$ curvature for $1 < \gamma < 2$. The basic theory for $P_{2\gamma}$ is developed by Chang-Gonzalez in their work [ChGon]. See Section 2 for more notations about that. The main part of this paper is to study the covering geometry of compact locally conformally flat manifold (M^n, g) with $R_g > 0$ and $Q_{2\gamma} > 0$ for $1 < \gamma < 2$. Specifically, the main theorem of this paper is the following estimate on the Hausdorff dimension of the limit set.

Theorem 1.1. *Let $n \geq 4$ and let (M^n, g) be a compact locally conformally flat manifold with $R_g > 0$ and $Q_{2\gamma} > 0$ for some $\gamma \in (1, 2)$, then*

$$\dim_{\mathcal{H}}(\Lambda) < \frac{n - 2\gamma}{2}. \quad (1.8)$$

Remark 1.1. Example 6.3 shows that the Hausdorff dimension upper bound (1.8) is optimal.

An immediate application of Theorem 1.1 is the following existence result for fractional order Yamabe problem.

Theorem 1.2. *Let (M^n, g) be a compact locally conformally flat manifold with $R_g > 0$. Assume that $Q_{2\gamma} > 0$ for some $\gamma \in (1, \max\{2, \frac{n}{2}\})$, then for every $\gamma' \in (1, \gamma]$, there exists a smooth Riemannian metric \hat{g} which is conformal to g such that $\hat{Q}_{2\gamma'} \equiv 1$.*

The above Hausdorff dimension estimate in effect leads to the following topological consequences. From now on, we assume that $\pi_1(M^n)$ is torsion-free (see the discussion at the end of Section 2 for the necessity of this assumption).

For $n = 3$, the following sphere theorem holds.

Theorem 1.3. *Let (M^3, g) be a compact conformally flat manifold with $R_g > 0$ and $Q_3 > 0$, then M^3 is conformal to (\mathbb{S}^3, g_1) with the round metric.*

When $n = 4$ or 5 , we have the following topological rigidity.

Theorem 1.4. *Given $n = 4$ or 5 , assume that (M^n, g) is a compact locally conformally flat manifold with $R_g > 0$ and $Q_{2\gamma} > 0$ for some $\gamma \in (1, 2)$, then M^n is diffeomorphic to \mathbb{S}^n or $\#_k(\mathbb{S}^1 \times \mathbb{S}^{n-1})$ for some $k \in \mathbb{Z}_+$ if one of the following holds:*

- (1) $n = 4, 1 < \gamma < 2$;
- (2) $n = 5, \frac{3}{2} \leq \gamma < 2$.

Remark 1.2. In Theorem 1.4, if $\pi_1(M^n)$ is non-elementary, then Γ is isomorphic to a Schottky group.

Remark 1.3. In case (2) of Theorem 1.4, the lower bound $\gamma \geq \frac{3}{2}$ is sharp. In fact, by Example 6.3, (M^5, ω) satisfies

$$R_g > 0, Q_{2\gamma} > 0, \quad (1.9)$$

for every $1 < \gamma < \frac{3}{2}$ and $Q_3 \equiv 0$, but clearly $M^5 = \mathbb{S}^3 \times \Sigma^2$ is not covered by $\#_k(\mathbb{S}^1 \times \mathbb{S}^4)$.

Proof of Theorem 1.4. First, we consider the case that $\Gamma \equiv \pi_1(M^n) \neq \{e\}$ is elementary. Since $R_g > 0$, it is a classical result by [GL] that M^n is not covered by \mathbb{T}^n . Therefore Γ has to be an infinite cyclic group generated by a loxodromic element, and hence M^n is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. So Theorem 1.4 holds in this case.

Now we switch to the general case in which $\Gamma \equiv \pi_1(M^n)$ is non-elementary. Under the assumption either (1) or (2), by Theorem 1.1, we have that $\dim_{\mathcal{H}}(\Lambda(\Gamma)) < 1$. Furthermore, Patterson-Sullivan's theorem (see theorem 2.7) implies that $\delta(\Gamma) = \dim_{\mathcal{H}}(\Lambda(\Gamma)) < 1$. Since Γ is non-elementary, $0 < \delta(\Gamma) < 1$. Applying theorem 6.1 in [Izeki], M^n is diffeomorphic to $\#_k(\mathbb{S}^1 \times \mathbb{S}^{n-1})$ for $k \in \mathbb{Z}_+$. \square

2. PRELIMINARIES

2.1. Basics in Kleinian Groups. We start with some brief review on the basic concepts about Kleinian groups and their useful properties in conformal geometry.

Let $\Gamma \leq \text{Conf}(\mathbb{S}^n) \cong \text{Isom}(\mathbb{B}^{n+1})$ be a Kleinian group which by definition gives a properly discontinuous action on a non-empty subdomain $\Omega(\Gamma) \subset \mathbb{S}^n$. The notion of limit set is fundamental in the study of Kleinian groups.

Definition 2.1 (Limit set). Let Γ be a Kleinian group, then the limit set is defined by

$$\Lambda(\Gamma) \equiv \{x \in (\mathbb{S}^n, g_1) \mid \exists \gamma_j \in \Gamma, y \in (\mathbb{S}^n, g_1), \text{ s.t. } \lim_{j \rightarrow \infty} d_{g_1}(\gamma_j(y), x) \rightarrow 0\}. \quad (2.1)$$

Immediately, by definition, $\Lambda(\Gamma)$ is a Γ -invariant closed subset in \mathbb{S}^n . Moreover, if Γ is co-compact, then

$$\Lambda(\Gamma) = \partial\Omega(\Gamma) = \mathbb{S}^n \setminus \Omega(\Gamma). \quad (2.2)$$

If Γ acts freely on $\Omega(\Gamma)$, then the quotient space $\Omega(\Gamma)/\Gamma$ is a locally conformally flat manifold. Schoen-Yau's result in [SY] proved that there are a large class of locally conformally flat manifolds of the form $\Omega(\Gamma)/\Gamma$. More precisely, let (M^n, g) be locally conformally flat, if $R_g \geq 0$, then the universal cover \widetilde{M}^n can be conformally embedded in \mathbb{S}^n .

Next we introduce the concept of Poincaré exponent and its applications in the study of conformal geometry.

Definition 2.2 (Poincaré exponent). Let Γ be a Kleinian group. Given $s > 0$, define the Poincaré series as follows,

$$\mathcal{P}_c(s; x) \equiv \sum_{\gamma \in \Gamma} \|\gamma'(x)\|^s, \quad (2.3)$$

where $\|\gamma'(x)\|$ is the length of the conformal factor. Poincaré exponent is defined by

$$\delta(\Gamma) \equiv \inf \left\{ s > 0 \mid \mathcal{P}_c(s; x) < \infty, \forall x \in \mathbb{S}^n \right\}. \quad (2.4)$$

There are several equivalent definitions of Poincaré exponent. Now we describe Poincaré exponent on a hyperbolic space $(\mathbb{B}^{n+1}, g_{-1}, 0^{n+1})$ with $\text{sec}_{g_{-1}} \equiv -1$.

Lemma 2.3. Let $\Gamma \leq \text{Isom}(\mathbb{B}^{n+1}) \cong \text{Conf}(\mathbb{S}^n)$ be a Kleinian group and denote by $\delta(\Gamma)$ the Poincaré exponent in terms of Definition 2.2, then the following holds:

(1) On $(\mathbb{B}^{n+1}, g_{-1}, 0^{n+1})$, let

$$\mathcal{P}_e(s; 0^{n+1}) \equiv \sum_{\gamma \in \Gamma} e^{-sd_{g_{-1}}(0^{n+1}, \gamma \cdot 0^{n+1})}, \quad (2.5)$$

then we have that

$$\delta(\Gamma) = \inf \left\{ s > 0 \mid \mathcal{P}_e(s; 0^{n+1}) < \infty \right\}. \quad (2.6)$$

(2) Let $n(R) \equiv \#\{\gamma \in \Gamma \mid d_{g_{-1}}(\gamma \cdot 0^{n+1}, 0^{n+1}) \leq R\}$, then

$$\delta(\Gamma) = \limsup_{R \rightarrow \infty} \frac{\log n(R)}{R}. \quad (2.7)$$

Example 2.1. Let Γ be elementary and torsion-free, namely, $|\Lambda(\Gamma)| \leq 2$. Then Γ must be an infinite cyclic group generated by a loxodromic element or a parabolic group of rank k for some $k \geq 1$. In the above two cases, $\delta(\Gamma) = 0, k/2$ respectively.

The above concept of Poicnaré exponent gives many interesting consequences in conformal geometry. To this end, we start with some analytic preliminaries. Let (M^n, g) be a closed locally conformally flat manifold and assume that \widetilde{M}^n is conformally embedded in \mathbb{S}^n . Therefore, there is some subdomain $\Omega(\Gamma) \subset \mathbb{S}^n$ such that $M^n = \Omega(\Gamma)/\Gamma$ where $\Gamma \cong \pi_1(M^n)$ is a Kleinian group.

By stereographic projection, the universal cover \widetilde{M}^n can be viewed as a subdomain in \mathbb{R}^n with the Riemannian covering metric $\tilde{g} = e^{2w}g_0$. Since $R_{g_0} \equiv 0$, the conformal change $g_0 = e^{-2w}\tilde{g}$ gives the following natural elliptic equation,

$$-\Delta_{\tilde{g}}(e^{-\frac{n-2}{2}w}) + \frac{n-2}{2}J_{\tilde{g}}e^{-\frac{n-2}{2}w} = 0, \quad (2.8)$$

where $J_{\tilde{g}} \equiv \frac{R_{\tilde{g}}}{2(n-1)}$. Cheng-Yau's gradient estimate gives that for every $x \in \mathbb{R}^n \setminus \Lambda$ with $B_{4R}(x) \subset \mathbb{R}^n \setminus \Lambda$,

$$\sup_{y \in B_R(x)} |\nabla_{\tilde{g}} \log e^{-\frac{n-2}{2}w(y)}| \leq C(n, \tilde{g}, R). \quad (2.9)$$

As a corollary, we have the following Harnack inequality, there exists $C(n, \tilde{g}, R) > 0$ such that for every $y, z \in B_R(x)$

$$C^{-1} \cdot e^{w(z)} \leq e^{w(y)} \leq C \cdot e^{w(z)}. \quad (2.10)$$

We give some basic properties of the conformal factors, which will be used frequently in the paper.

Lemma 2.4 ([CQY]). *If $M^n \cong \Omega(\Gamma)/\Gamma$ is closed, then there exists $K > 0$ such that for every $x \in \Omega(\Gamma)$,*

$$K^{-1} \cdot d_0^{-1}(x, \Lambda) \leq e^{w(x)} \leq K \cdot d_0^{-1}(x, \Lambda), \quad (2.11)$$

where d_0 is the distance function with respect to Euclidean metric.

Next we give an integrability lemma for the conformal factor e^w .

Lemma 2.5. *Let Γ be a Kleinian group and denote by $\delta_c \equiv \delta(\Gamma) \geq 0$, then $e^w \in L_{loc}^p(\mathbb{R}^n)$ for every $0 < p < n - \delta_c$.*

Proof. Let $\psi : \mathbb{S}^n \setminus \{N\} \rightarrow \mathbb{R}^n$ be the stereographic projection. First, we show that for every $0 < p < n - \delta_c$, $e^{\tilde{w}} \equiv e^{w \circ \psi} \cdot \left(\frac{1+|x|^2}{2}\right) \in L^p(\mathbb{S}^n)$. In fact, for every $\gamma \in \Gamma$ and for every $\tilde{x} \in \mathbb{S}^n \setminus \Lambda$,

$$|\gamma'(\tilde{x})| \cdot e^{\tilde{w}(\gamma \cdot \tilde{x})} = e^{\tilde{w}(\tilde{x})}. \quad (2.12)$$

Let $K \subset \Omega(\Gamma)$ be a bounded subset such that $F \subset K$. By (2.13), there exists some positive constant $C_0(\tilde{g}, \text{diam}_{\tilde{g}}(K), n) > 0$ such that that for every $\tilde{x}, \tilde{y} \in K$,

$$C_0^{-1} \cdot e^{\tilde{w}(\tilde{x})} \leq e^{\tilde{w}(\tilde{y})} \leq C_0 \cdot e^{\tilde{w}(\tilde{x})}. \quad (2.13)$$

Now fix a fundamental domain $F \subset \Omega(\Gamma)$, then the integral over \mathbb{S}^n can be reduced to the following sum,

$$\begin{aligned} \int_{\Omega(\Gamma)} e^{p \cdot w(\tilde{x})} dv_{g_1}(\tilde{x}) &= \sum_{\gamma \in \Gamma} \int_{\gamma \cdot F} e^{p \cdot w(\gamma \cdot \tilde{x})} dv_{g_1}(\gamma \cdot \tilde{x}) \\ &= \sum_{\gamma \in \Gamma} \int_F |\gamma'(\tilde{x})|^{n-p} \cdot e^{p \cdot w(\tilde{x})} dv_{g_1}(\tilde{x}). \end{aligned} \quad (2.14)$$

Harnack inequality (2.13) implies that for fixed $\tilde{x}_0 \in F$, we have that for every $\tilde{x} \in F$,

$$e^{\tilde{w}(\tilde{x})} \leq C_0 \cdot e^{\tilde{w}(\tilde{x}_0)}. \quad (2.15)$$

Therefore,

$$\int_{\Omega(\Gamma)} e^{p \cdot w(\tilde{x})} dv_{g_1}(\tilde{x}) \leq C_0^p \cdot \text{Vol}_{\tilde{g}}(F) \cdot \sum_{\gamma \in \Gamma} |\gamma'(\tilde{x}_0)|^{n-p}. \quad (2.16)$$

Since $p < n - \delta_c$, by the definition of Poincaré exponent, the above integral is finite.

$$\int_E e^{p \cdot w(x)} dv_0(x) = \int_{\psi^{-1}(E)} e^{p \cdot \tilde{w}(\tilde{x})} \cdot \left(\frac{1+|x|^2}{2}\right)^{p+n} dv_1(\tilde{x}) < \infty. \quad (2.17)$$

□

Now we discuss the measure-theoretic properties of Kleinian groups.

Definition 2.6 (Hausdorff Content and Hausdorff Dimension). Let $S \subset \mathbb{R}^n$ and $d \in [0, +\infty)$, then the d -dimensional Hausdorff content of S is defined by

$$C_{\mathcal{H}}^d(S) \equiv \inf \left\{ \sum_i r_i^d : \text{there is a cover of } S \text{ by balls with radii } r_i > 0 \right\}. \quad (2.18)$$

The Hausdorff dimension of S is defined by

$$\dim_{\mathcal{H}}(S) \equiv \inf \{d \geq 0 | C_{\mathcal{H}}^d(S) = 0\}. \quad (2.19)$$

If Γ is non-elementary, Patterson-Sullivan characterized Poincaré exponent by the Hausdorff dimension of the limit set $\Lambda(\Gamma)$.

Theorem 2.7 (Patterson-Sullivan). *Let Γ be a non-elementary Kleinian group, then*

$$\delta(\Gamma) = \dim_{\mathcal{H}}(\Lambda(\Gamma)). \quad (2.20)$$

The following lemma is rather standard and it is the foundation of the Hausdorff dimension estimate in our paper. The proof is given in the paper [CHY].

Lemma 2.8. *Let $K \subset \mathbb{R}^n$ be compact. Denote by $d_0(x, K) \equiv \inf\{|x - y| : y \in K\}$ for $x \in \mathbb{R}^n$. Assume that for some $R > 0$ and $\alpha \geq 1$, we have $K \subset B_R(0^n)$ and*

$$\int_{B_R(0^n) \setminus K} d_0(x, K)^{-\alpha} dx < \infty, \quad (2.21)$$

then $\dim_{\mathcal{H}}(K) \leq n - \alpha$. In addition, if $\alpha \geq n$, then $K = \emptyset$.

Notice that, with Kleinian group action, Lemma 2.5 is a converse of lemma 2.8.

Now we end this section by stating a classical theorem by Schoen-Yau ([SY]), which has been mentioned in Section 1.

Theorem 2.9. *Let (M^n, g) be a closed locally conformally flat manifold with $R_g \geq 0$, then its universal \widetilde{M}^n can be conformally embedded in \mathbb{S}^n . Furthermore, $\dim_{\mathcal{H}}(\Lambda(\Gamma)) \leq \frac{n-2}{2}$, where $\Gamma \cong \pi_1(M^n)$ and $\Lambda(\Gamma)$ is the limit set on \mathbb{S}^n .*

Remark 2.1. In the above theorem, if the assumption $R_g \geq 0$ is replaced with $R_g > 0$, then correspondingly we have $\dim_{\mathcal{H}}(\Lambda(\Gamma)) < \frac{n-2}{2}$.

2.2. Fractional GJMS Operator and Fractional Q Curvature. Applying scattering theory on conformally compact Einstein manifolds developed by Graham-Zworski ([GrZ]), Chang-González defined the notation of fractional GJMS operator in their joint paper [ChGon]. In this section, we briefly review the definition of fractional GJMS operator and the corresponding fractional Q curvature. We start with the basic notions for defining fractional GJMS operators. Let (X^{n+1}, M^n, g_+) be a conformally compact Einstein manifold with conformal infinity $(M^n, [h])$ which satisfies the following:

- (1) $M^n = \partial X^{n+1}$,
- (2) g_+ is complete on $\text{Int}(X^{n+1})$ such that $\text{Ric}_{g_+} \equiv -ng_+$,
- (3) there exists a smooth defining function of M^n , denoted by ρ , such that $\rho^2 g_+$ is compact on $\overline{X^{n+1}}$ with $\rho^2 g_+|_{TM^n} \in [h]$ and

$$\begin{cases} \rho(x) > 0, & x \in \text{Int}(X^{n+1}), \\ \rho(x) = 0, & x \in M^n, \\ |\nabla \rho|(x) \neq 0, & x \in M^n. \end{cases} \quad (2.22)$$

The following fact is standard (see [Gr] or [Lee]).

Lemma 2.10 (Geodesic defining function). *Let $(M^n, [h])$ be the conformal infinity, then for every $h_0 \in [h]$ there exists a unique defining function r in a neighborhood of M^n such that*

- (1) $r^2 g_+|_{r=0} = h_0$,
- (2) *there exists $\epsilon > 0$ such that $|\nabla r|_{r^2 g_+} \equiv 1$ on $M^n \times [0, \epsilon)$.*

Now we are in a position to define fractional GJMS operators (see [ChGon] for more details). According to the above notations, let $s \in \mathbb{C} \setminus \{\frac{n}{2}\}$, we will consider the following Poisson equation

$$(\Delta_{g_+} + s(n-s))u = 0 \quad (2.23)$$

and consider its generalized eigenfunction

$$u = Fy^{n-s} + Gy^s, \quad F, G \in C^\infty(X^{n+1}), \quad (2.24)$$

where y is the geodesic defining function for a given boundary (M^n, h_0) with $h_0 \in [h]$ (see lemma 2.10). The scattering operator $S(s)$ is a Dirichlet-to-Neumann operator which is given by the following: if $f \equiv F|_{y=0}$, then

$$S(s)f \equiv G|_{y=0}. \quad (2.25)$$

In [GrZ], it was proved that $S(s)$ is a meromorphic family conformally covariant operators for s with simple poles in $\frac{n}{2} + \mathbb{Z}_+$. Now given $\gamma \in (0, \frac{n}{2})$, the fractional GJMS operator $P_{2\gamma}$ is defined by

$$P_{2\gamma}f \equiv P_{2\gamma}[h_0, g_+]f \equiv 2^{2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} S(\frac{n}{2} + \gamma)f. \quad (2.26)$$

Notice that the function $\Gamma(\gamma)$ cancels the simple poles of the scattering operator $S(\frac{n}{2} + \gamma)$, and thus the fractional GJMS operator $P_{2\gamma}$ is continuous when $\gamma \in (0, \frac{n}{2})$. Moreover, if $\gamma \in (0, n/2)$, the operator $P_{2\gamma}$ satisfies the following conformal covariance property: let $\hat{h} = v^{\frac{4}{n-2\gamma}}h$ and let $\hat{P}_{2\gamma}$ be the fractional GJMS operator with respect to \hat{h} , then

$$\hat{P}_{2\gamma}(u) = v^{-\frac{n+2\gamma}{n-2\gamma}} P_{2\gamma}(uv). \quad (2.27)$$

for any $u \in C^\infty(M^n)$. With the fractional GJMS operator $P_{2\gamma}$, we will define fractional Q curvature. We start with a simpler case $0 < \gamma < \frac{n}{2}$. Let $P_{2\gamma}$ be in (2.26), then the corresponding fractional Q curvature is defined by

$$Q_{2\gamma} \equiv (\frac{n-2\gamma}{2})^{-1} P_{2\gamma}(1). \quad (2.28)$$

In Section 5, we will discuss the critical case where $n = 3$ and $\gamma = \frac{3}{2}$. Actually, in general, we can modify (2.26) and (2.28) to define the operator $P_{2\gamma}$ and $Q_{2\gamma}$ curvature in the critical case $\gamma = \frac{n}{2}$. First, if n is even and $\gamma = \frac{n}{2} \equiv k \in \mathbb{Z}_+$, then $s = n$ is a simple pole of the scattering operator $S(s)$, while the function $\Gamma(-\gamma)$ has a simple pole at $\gamma = \frac{n}{2}$, so we define

$$P_{2k} \equiv c_k \cdot \text{Res}_{s=n} S(s), \quad (2.29)$$

where $c_k \equiv (-1)^{k+1} 2^{2k} k! (k-1)!$. Correspondingly, the curvature Q_n is given by

$$Q_n = c_k \cdot S(n)1. \quad (2.30)$$

Notice that, if n is even, $S(s)1$ is holomorphic when $s = n$. Now assume that n is odd, then the operator $S(\frac{n}{2} + \gamma)$ is continuous at $\gamma = \frac{n}{2}$ and $\Gamma(-\gamma)$ is also continuous at $\gamma = \frac{n}{2}$. In this case, the definition of $P_{2\gamma}$ is identical to that in the case $\gamma < \frac{n}{2}$. By [GrZ], if n is odd, $\lim_{s \rightarrow n} S(s)1 = 0$, then the limit $\lim_{\gamma \rightarrow n/2} (\frac{n-2\gamma}{2})^{-1} P_{2\gamma}(1)$ exists and so we define

$$Q_n \equiv \lim_{\gamma \rightarrow n/2} (\frac{n-2\gamma}{2})^{-1} P_{2\gamma}(1). \quad (2.31)$$

By the computations in [GrZ], when $\gamma = \frac{n}{2}$, the following conformal covariance property always holds (n is either even or odd): if $\hat{g} = e^{2w}g$ then,

$$e^{nw} \hat{Q}_n = Q_n + P_n(w). \quad (2.32)$$

We end this section by giving a remark about the definition of the fractional GJMS operator. Given a conformally compact Einstein manifold (X^{n+1}, g_+, M^n) with a boundary (M^n, h_0) , then by definition, the operator $P_{2\gamma} \equiv P_{2\gamma}[h_0, g_+]$ depends not only on the boundary geometry (M^n, h_0) but also on the global geometry (X^{n+1}, g_+) . In other words, given (M^n, h) , the definition of the operator $P_{2\gamma}$ on M^n depends on the choice of the space (X^{n+1}, g_+) . In our paper, we focus on the manifolds (M^n, g) which are locally conformally flat with $R_g > 0$. Notice that, in this setting, (M^n, g) is not covered by a torus and thus it is a Kleinian manifold with $\Gamma \cong \pi_1(M^n) \leq \text{Conf}(\mathbb{S}^n)$ convex co-compact. Furthermore, if Γ is torsion-free, then $(M^n, [g])$ can be viewed as the conformal infinity of the complete hyperbolic manifold \mathbb{H}^{n+1}/Γ (otherwise, \mathbb{H}^{n+1}/Γ is a hyperbolic orbifold with finite singularities). Throughout this paper, we will assume that $\pi_1(M^n)$ is *torsion-free* which is necessary for the background of scattering theory from the above discussion, and the fractional GJMS operator $P_{2\gamma}$ in this paper is always defined by the hyperbolic filling-in in the above context.

3. FRACTIONAL LAPLACIANS ON EUCLIDEAN SPACE

We reviewed in Section 2.2 the definition of GJMS operators on a conformally compact Einstein manifold. In this section, as a special case, we will focus on those operators on Euclidean space with hyperbolic filling-in which plays a crucial role in the proof of the main theorems of this paper.

3.1. Harmonic Extension and Equivalent Definitions of Fractional Laplacians. Let $\alpha \in (0, 1)$, first we introduce the following space

$$L_{2\alpha}(\mathbb{R}^n) \equiv \left\{ u : \mathbb{R}^n \rightarrow \mathbb{R}^1 \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2\alpha}} dx < \infty \right\} \quad (3.1)$$

which is a natural space in the definition of fractional Laplacian. Now we briefly review Caffarelli-Silvestre's result about harmonic extensions in [CaSi]. For fixed $\alpha \in (0, 1)$, denote by $b \equiv 1 - 2\alpha \in$

$(-1, 1)$, the for every $f \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, there exists a unique function $U(x, y) \in L^\infty(\mathbb{R}_+^{n+1})$ such that

$$\begin{cases} \operatorname{Div}(y^b \nabla U)(x, y) \equiv 0. \\ U(x, 0) = f(x). \end{cases} \quad (3.2)$$

Moreover, $U(x, y)$ can be expressed as the following Poisson integral

$$U(x, y) \equiv C_{n,\alpha} \int_{\mathbb{R}^n} \frac{y^{1-b} \cdot f(\xi)}{(|x - \xi|^2 + y^2)^{\frac{n+1-b}{2}}} d\xi. \quad (3.3)$$

In fact, the above constant $C_{n,\alpha} > 0$ is chosen such that

$$C_{n,\alpha} \int_{\mathbb{R}^n} \frac{y^{1-b}}{(|x - \xi|^2 + y^2)^{\frac{n+1-b}{2}}} d\xi = 1. \quad (3.4)$$

With the above harmonic extension, Caffarelli-Silvestre proved that the definition of fractional Laplacian given by harmonic extension (Definition 3.1) is actually equivalent to the one defined in terms of Fourier transform (Definition 3.2) and equivalent to the one defined in terms of singular integral (Definition 3.3) as well.

Definition 3.1. Let $d_{n,\alpha} > 0$ be some positive constant depending only on n and α , then we define

$$(-\Delta)^\alpha f(x) \equiv d_{n,\alpha} \cdot \lim_{y \rightarrow 0+} -y^b \frac{\partial U}{\partial y}. \quad (3.5)$$

Definition 3.2. Let $f \in L_{2\alpha}(\mathbb{R}^n)$ which is viewed as a tempered distribution, then the fractional Laplacian $(-\Delta)^\alpha$ is given by the following equation,

$$\mathcal{F}((-\Delta)^\alpha f) \equiv |\xi|^{2\alpha} \mathcal{F}(f). \quad (3.6)$$

Definition 3.3. Let $d_{n,\alpha} > 0$ be some positive constant, then we define

$$(-\Delta)^\alpha f(x) \equiv d_{n,\alpha} \cdot \text{P. V.} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+2\alpha}} dy, \quad (3.7)$$

where $f \in L_{2\alpha}(\mathbb{R}^n)$.

In fact, the above definitions are equivalent to the following definition in terms of second order difference.

Lemma 3.4. *There exists some positive constant $d_{n,\alpha} > 0$ such that*

$$(-\Delta)^\alpha f(x) = -d_{n,\alpha} \cdot \int_{\mathbb{R}^n} \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{n+2\alpha}} dy, \quad (3.8)$$

where $f \in L_{2\alpha}(\mathbb{R}^n)$.

The following lemma is straightforward.

Lemma 3.5. *Given $f \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, let $U_1(x, y)$ be the harmonic extension of f^{τ_1} for some $\tau_1 > 0$ and let $U_2(x, y)$ be the harmonic extension of f^{τ_2} for some $\tau_2 > \tau_1 > 0$. Then for every $(x, y) \in \mathbb{R}_+^{n+1}$, we have that*

$$(U_1)^{\tau_2/\tau_1}(x, y) \leq U_2(x, y). \quad (3.9)$$

Proof. Equivalently, we will prove that for every $(x, y) \in \mathbb{R}_+^{n+1}$

$$U_1(x, y) \leq (U_2)^{\tau_1/\tau_2}(x, y). \quad (3.10)$$

By Poisson's formula, we have the following,

$$U_1(x, y) = C_{n,\alpha} \int_{\mathbb{R}^n} \frac{y^{1-b} \cdot f^{\tau_1}(\xi)}{(|x - \xi|^2 + y^2)^{\frac{n+1-b}{2}}} d\xi, \quad (3.11)$$

and

$$U_2(x, y) = C_{n,\alpha} \int_{\mathbb{R}^n} \frac{y^{1-b} \cdot f^{\tau_2}(\xi)}{(|x - \xi|^2 + y^2)^{\frac{n+1-b}{2}}} d\xi. \quad (3.12)$$

In fact,

$$\begin{aligned} U_1(x, y) &= C_{n,\alpha} \int_{\mathbb{R}^n} \frac{y^{1-b} \cdot f^{\tau_1}(\xi)}{(|x - \xi|^2 + y^2)^{\frac{n+1-b}{2}}} d\xi \\ &= C_{n,\alpha} \left(\int_{\mathbb{R}^n} \left(\frac{y^{1-b}}{(|x - \xi|^2 + y^2)^{\frac{n+1-b}{2}}} \right)^{1-\frac{\tau_1}{\tau_2}} \cdot \left(\frac{y^{1-b} f^{\tau_2}(\xi)}{(|x - \xi|^2 + y^2)^{\frac{n+1-b}{2}}} \right)^{\frac{\tau_1}{\tau_2}} d\xi \right) \\ &\stackrel{\text{H\"older}}{\leq} C_{n,\alpha} \left(\int_{\mathbb{R}^n} \frac{y^{1-b}}{(|x - \xi|^2 + y^2)^{\frac{n+1-b}{2}}} d\xi \right)^{1-\frac{\tau_1}{\tau_2}} \left(\int_{\mathbb{R}^n} \frac{y^{1-b} f^{\tau_2}(\xi)}{(|x - \xi|^2 + y^2)^{\frac{n+1-b}{2}}} d\xi \right)^{\frac{\tau_1}{\tau_2}} \\ &= \left(C_{n,\alpha} \int_{\mathbb{R}^n} \frac{y^{1-b}}{(|x - \xi|^2 + y^2)^{\frac{n+1-b}{2}}} d\xi \right)^{1-\frac{\tau_1}{\tau_2}} \left(C_{n,\alpha} \int_{\mathbb{R}^n} \frac{y^{1-b} f^{\tau_2}(\xi)}{(|x - \xi|^2 + y^2)^{\frac{n+1-b}{2}}} d\xi \right)^{\frac{\tau_1}{\tau_2}} \\ &\stackrel{(3.4)}{=} \left(C_{n,\alpha} \int_{\mathbb{R}^n} \frac{y^{1-b} f^{\tau_2}(\xi)}{(|x - \xi|^2 + y^2)^{\frac{n+1-b}{2}}} d\xi \right)^{\frac{\tau_1}{\tau_2}} \\ &= (U_2)^{\tau_1/\tau_2}(x, y). \end{aligned} \quad (3.13)$$

□

3.2. Comparison Principle for Fractional Laplacians. Graham and Zworski developed the deep connection between scattering theory and conformal geometry in their fundamental work [GrZ]. The following proposition is a crucial observation in the proof of the main theorems. The proof follows from the combination of Lemma 3.5 and the expansion (2.24).

Proposition 3.6. *Given $0 < \alpha < 1$ and $\tau_2 > \tau_1 > 0$. Let $f \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ then for every $x \in \mathbb{R}^n$,*

$$\left((-\Delta)^\alpha f^{\tau_1} \right)(x) \geq \frac{\tau_1}{\tau_2} \cdot f^{\tau_1-\tau_2}(x) \cdot \left((-\Delta)^\alpha f^{\tau_2} \right)(x). \quad (3.14)$$

Proof. Let U_1 be the harmonic extension of f^{τ_1} and let U_2 be the harmonic extension of f^{τ_2} . On the other hand, let $s \equiv \frac{n+2\alpha}{2}$ and immediately $2s - n \in (0, 2)$. Now denote

$$u_1 \equiv y^{n-s} \cdot U_1, \quad u_2 \equiv y^{n-s} \cdot U_2, \quad (3.15)$$

and then by the calculations in [ChGon], we have that

$$\Delta_{\mathbb{H}^{n+1}} u_1 + s(n-s)u_1 = 0, \quad \Delta_{\mathbb{H}^{n+1}} u_2 + s(n-s)u_2 = 0. \quad (3.16)$$

Hence, we obtain the Taylor expansion of U_1 in geodesic defining function near to the slice $\{y = 0\}$:

$$U_1 = \left(f^{\tau_1} + O(y^2)\right) + \left((S(s)f^{\tau_1}) \cdot y^{2s-n} + O(y^{2s-n+2})\right), \quad (3.17)$$

and

$$U_2 = \left(f^{\tau_2} + O(y^2)\right) + \left((S(s)f^{\tau_2}) \cdot y^{2s-n} + O(y^{2s-n+2})\right). \quad (3.18)$$

Notice that $2s - n \in (0, 2)$, so near to the boundary $\{y = 0\}$, it holds that

$$\begin{aligned} (U_1)^{\tau_2/\tau_1} &= \left(f^{\tau_1} + (S(s)f^{\tau_1}) \cdot y^{2s-n} + O(y^2)\right)^{\tau_2/\tau_1} \\ &= f^{\tau_2} \cdot \left(1 + \frac{\tau_2}{\tau_1} \cdot \frac{S(s)f^{\tau_1}}{f^{\tau_1}} \cdot y^{2s-n} + O(y^2)\right), \end{aligned} \quad (3.19)$$

and

$$U_2 = f^{\tau_2} \cdot \left(1 + \frac{S(s)f^{\tau_2}}{f^{\tau_2}} \cdot y^{2s-n} + O(y^2)\right). \quad (3.20)$$

By Lemma 3.5, it holds that $(U_1)^{\tau_2/\tau_1} \leq U_2$, which implies that

$$\frac{\tau_2}{\tau_1} \cdot \frac{S(s)f^{\tau_1}}{f^{\tau_1}} \leq \frac{S(s)f^{\tau_2}}{f^{\tau_2}}. \quad (3.21)$$

By definition, $(-\Delta)^\alpha \equiv 2^{2\alpha} \cdot \frac{\Gamma(\alpha)}{\Gamma(-\alpha)} \cdot S(s)$ with $s = \frac{n+2\alpha}{2}$, and thus

$$\frac{\tau_2}{\tau_1} \cdot \frac{(-\Delta)^\alpha f^{\tau_1}}{f^{\tau_1}} \geq \frac{(-\Delta)^\alpha f^{\tau_2}}{f^{\tau_2}}. \quad (3.22)$$

Therefore,

$$((-\Delta)^\alpha f^{\tau_1})(x) \geq \frac{\tau_1}{\tau_2} \cdot f^{\tau_1-\tau_2} \left((-\Delta)^\alpha f^{\tau_2} \right)(x), \quad \forall x \in \mathbb{R}^n. \quad (3.23)$$

□

3.3. Smooth Approximation and Convergence Lemmas. The Hausdorff dimension estimate in Theorem 1.1 relies on delicate estimates on the conformal factor e^w . A crucial point to see this is to apply Proposition 3.6 to the conformal factor e^w . However, Proposition 3.6 works only for the globally smooth function in \mathbb{R}^n , while in our context, the function e^w blows up along the limit set $\Lambda(\Gamma)$ (see Lemma 2.4). So it is necessary to approximate e^w by a family of converging smooth functions and obtain the comparison for the smooth approximations. The key step in this procedure is to prove the comparison inequalities of the smoothing approximations indeed converge to the comparison inequality for e^w . So the primary goal of this section is to define the smooth approximations and prove various convergence results.

Let us start with the definition of the standard mollifier. Given $p \geq 1$, for every $f \in L_{loc}^p(\mathbb{R}^n)$, define

$$f_\epsilon(x) \equiv f * \eta_\epsilon(x) = \int_{\mathbb{R}^n} f(x-y)\eta_\epsilon(y)dy, \quad (3.24)$$

where

$$\eta_\epsilon(x) \equiv \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right) \geq 0, \quad (3.25)$$

satisfies the following

$$\eta \in C_0^\infty(\mathbb{R}^n), \text{ supp}(\eta) \in B_1(0^n), \int_{\mathbb{R}^n} \eta = 1. \quad (3.26)$$

Immediately, by the symmetry property of convolution, it holds that

$$f_\epsilon(x) = \int_{\mathbb{R}^n} f(x-y)\eta_\epsilon(y)dy = \int_{B_\epsilon(0^n)} f(x-y)\eta_\epsilon(y)dy. \quad (3.27)$$

Now we present some basic properties of the standard mollifiers which will be frequently used in the later proofs of the paper.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^n$ be an open subset. If $f \in C(\Omega)$, then for every compact subset $K \subset \Omega$, it holds that when $\epsilon \rightarrow 0$,*

$$\|f_\epsilon - f\|_{C^0(K)} \rightarrow 0. \quad (3.28)$$

The proof of Lemma 3.7 is straightforward and standard, so we omit the proof.

Next we introduce the convergence property of derivatives of mollifiers.

Lemma 3.8. *Let $f \in C^2(B_R(p))$, then for every $x \in B_R(p)$, it holds that:*

(1) *for every $1 \leq j \leq n$,*

$$\lim_{\epsilon \rightarrow 0} \frac{\partial f_\epsilon}{\partial x_j}(x) = \frac{\partial f}{\partial x_j}(x), \quad (3.29)$$

(2) *for every $1 \leq i \leq j \leq n$,*

$$\lim_{\epsilon \rightarrow 0} \frac{\partial^2 f_\epsilon}{\partial x_j \partial x_i}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x). \quad (3.30)$$

Proof. The proof of the lemma is standard. So we just give a quick outline of the proof of (1). Let $e_j \equiv \frac{\partial}{\partial x_j}$, then by mean value theorem,

$$\begin{aligned} \frac{f_\epsilon(x + he_j) - f_\epsilon(x)}{h} &= \int_{B_\epsilon(0^n)} \frac{f(x + he_j - y) - f(x - y)}{h} \cdot \eta_\epsilon(y) dy \\ &= \int_{B_\epsilon(0^n)} \frac{\partial f}{\partial x_j}(x - y + \xi(h)e_j) \cdot \eta_\epsilon(y) dy, \end{aligned} \quad (3.31)$$

where $0 \leq \xi(h) \leq h$. Applying dominated convergence theorem,

$$\begin{aligned} \frac{\partial f_\epsilon}{\partial x_j}(x) &= \lim_{h \rightarrow 0} \frac{f_\epsilon(x + he_j) - f_\epsilon(x)}{h} \\ &= \int_{B_\epsilon(0^n)} \frac{\partial f}{\partial x_j}(x - y) \cdot \eta_\epsilon(y) dy. \end{aligned} \quad (3.32)$$

So it follows that

$$\lim_{\epsilon \rightarrow 0} \frac{\partial f_\epsilon}{\partial x_j}(x) = \frac{\partial f}{\partial x_j}(x). \quad (3.33)$$

□

Lemma 3.9. *If $\lim_{|x| \rightarrow +\infty} f(x) \rightarrow 0$, then for every $\epsilon > 0$, $\lim_{|x| \rightarrow +\infty} f_\epsilon(x) \rightarrow 0$.*

Proof.

$$f_\epsilon(x) = \int_{B_\epsilon(0^n)} f(x - y) \eta_\epsilon(y) dy. \quad (3.34)$$

Let $g_{\epsilon,x}(y) \equiv f(x - y) \eta_\epsilon(y)$. Clearly, for every $y \in B_\epsilon(0^n)$,

$$\lim_{|x| \rightarrow +\infty} g_{\epsilon,x}(y) \rightarrow 0. \quad (3.35)$$

Moreover, $|g_{\epsilon,x}(y)| \leq \sup_{z \in B_\epsilon(x)} |f(z)| \cdot \eta_\epsilon(y) \leq \frac{C(n)}{\epsilon^n} \cdot \|f\|_{L^\infty(\mathbb{R}^n)}$. Then dominated convergence theorem implies that

$$\lim_{|x| \rightarrow +\infty} f_\epsilon(x) = \lim_{|x| \rightarrow +\infty} \int_{B_\epsilon(0^n)} g_{\epsilon,x}(y) dy = \int_{B_\epsilon(0^n)} \lim_{|x| \rightarrow +\infty} g_{\epsilon,x}(y) dy = 0. \quad (3.36)$$

□

The following convergence property of mollifiers is useful in the proof of convergence properties of fractional Laplacians.

Lemma 3.10. *Fix $\alpha \in (0, 1)$ and $x_0 \in \mathbb{R}^n$, and let $\Omega \equiv \mathbb{R}^n \setminus B_1(x_0) \subset \mathbb{R}^n$ be open. Assume that $f \in L^1(\Omega, \mu_{x_0})$ with $d\mu_{x_0}(y) = \frac{dy}{|x_0 - y|^{n+2\alpha}}$, then we have the following:*

(1) *There exists $C_0(n) > 0$ such that for every $0 < \epsilon < 10^{-3}$, it holds that*

$$\int_{\Omega} |f_\epsilon| d\mu_{x_0} \leq C_0(n) \int_{\tilde{\Omega}} |f| d\mu_{x_0}, \quad (3.37)$$

where $\tilde{\Omega} \equiv \mathbb{R}^n \setminus B_{1/2}(x_0)$.

(2) When $\epsilon \rightarrow 0$,

$$\int_{\Omega} |f_{\epsilon} - f| d\mu_{x_0} \rightarrow 0. \quad (3.38)$$

Proof. (1) For every $\epsilon > 0$, by the definition of mollifiers and Fubini's theorem,

$$\begin{aligned} \int_{\Omega} |f_{\epsilon}(y)| d\mu_{x_0}(y) &= \int_{\Omega} \left| \int_{\mathbb{R}^n} f(y-z) \eta_{\epsilon}(z) dz \right| d\mu_{x_0}(y) \\ &\leq \int_{\Omega} \int_{B_{\epsilon}(0^n)} |f(y-z)| \eta_{\epsilon}(z) dz d\mu_{x_0}(y) \\ &= \int_{B_{\epsilon}(0^n)} \left(\int_{\Omega} |f(y-z)| d\mu_{x_0}(y) \right) \eta_{\epsilon}(z) dz. \end{aligned} \quad (3.39)$$

Since $z \in B_{\epsilon}(0^n)$, by triangle inequality, $\xi \equiv y - z \in \tilde{\Omega} \equiv \mathbb{R}^n \setminus B_{1/2}(x_0)$ if $y \in \Omega \equiv \mathbb{R}^n \setminus B_1(0^n)$. Moreover,

$$\int_{\Omega} |f(y-z)| d\mu_{x_0}(y) = \int_{\Omega} \frac{|f(y-z)| dy}{|x_0 - y|^{n+2\alpha}} \leq \int_{\tilde{\Omega}} \frac{|f(\xi)| d\xi}{|x_0 - \xi - z|^{n+2\alpha}} \leq C_0(n) \int_{\tilde{\Omega}} \frac{|f(\xi)| d\xi}{|x_0 - \xi|^{n+2\alpha}}. \quad (3.40)$$

Plugging into (3.39),

$$\begin{aligned} \int_{\Omega} |f_{\epsilon}| d\mu_{x_0} &\leq C_0(n) \int_{B_{\epsilon}(0^n)} \left(\int_{\tilde{\Omega}} |f(\xi)| d\mu_{x_0}(\xi) \right) \eta_{\epsilon}(z) dz \\ &\leq C_0(n) \int_{\tilde{\Omega}} |f| d\mu_{x_0}. \end{aligned} \quad (3.41)$$

So the proof of (1) is done.

(2) We will show that for every $\delta > 0$, there exists $\epsilon_0(\delta, n, f) > 0$ such that for every $\epsilon < \epsilon_0$, it holds that

$$\int_{\Omega} |f_{\epsilon}(y) - f(y)| d\mu_{x_0}(y) < \delta. \quad (3.42)$$

In fact, for every $\delta > 0$, there exists $g \in C_0(\Omega)$ with $\mu_{x_0}(\text{Supp}(g)) < C_1(n, \delta, f)$ such that

$$\int_{\tilde{\Omega}} |f - g| d\mu_{x_0} \leq \frac{\delta}{10C_0(n)}, \quad (3.43)$$

where $C_0(n) > 0$ is the constant in (1). Since $g \in C_0(\tilde{\Omega})$ with $\mu_{x_0}(\text{Supp}(g)) < C_1(n, \delta, f)$, so there exists some compact subset $K \subset T_{10^{-2}}(\text{Supp}(g)) \subset \tilde{\Omega}$ such that for every $0 < \epsilon < 10^{-3}$ we have $\text{Supp}(g_{\epsilon}) \subset T_{10^{-2}}(\text{Supp}(g))$. Denote by $M_1(\delta, n, f) \equiv \mu_{x_0}(T_{10^{-2}}(\text{Supp}(g))) > 0$, by Lemma 3.7, then for every $\delta > 0$ there exists $\epsilon_0(\delta, n, f) > 0$ such that for every $0 < \epsilon < \epsilon_0$, it holds that

$$\|g_{\epsilon} - g\|_{C^0(\text{Supp}(g))} < \frac{\delta}{10M_1}, \quad (3.44)$$

which implies that

$$\int_{\Omega} |g_{\epsilon} - g| d\mu_{x_0} \leq \int_{\tilde{\Omega}} |g_{\epsilon} - g| d\mu_{x_0} \leq \int_{T_{10-2}(\text{Supp}(g))} |g_{\epsilon} - g| d\mu_{x_0} < \frac{\delta}{10}. \quad (3.45)$$

Therefore, by (3.37), (3.43) and (3.44), for each $0 < \epsilon < \epsilon_0$,

$$\begin{aligned} \int_{\Omega} |f_{\epsilon} - f| d\mu_{x_0} &\leq \int_{\Omega} |f_{\epsilon} - g_{\epsilon}| d\mu_{x_0} + \int_{\Omega} |g_{\epsilon} - g| d\mu_{x_0} + \int_{\Omega} |g - f| d\mu_{x_0} \\ &\leq C_0(n) \cdot \int_{\tilde{\Omega}} |f - g| d\mu_{x_0} + \frac{\delta}{10} + \frac{\delta}{10C_0(n)} \\ &< \frac{\delta}{10} + \frac{\delta}{10} + \frac{\delta}{10C_0(n)} \\ &< \delta. \end{aligned} \quad (3.46)$$

So the proof of (2) is done. □

With the above basic properties of mollifiers, now we focus on the convergence properties of fractional Laplacian $(-\Delta)^{\alpha}$ acting on the standard mollifiers f_{ϵ} . In the following proofs, mainly we will choose the definition of $(-\Delta)^{\alpha}$ in terms of second order difference (see Definition 3.4).

Let $\Lambda \subset \mathbb{R}^n$ be a closed subset of zero measure. From now on, we will study functions which satisfy the following assumption:

Assumption 3.11. $f \in L_{2\alpha}(\mathbb{R}^n) \cap C^{\infty}(\mathbb{R}^n \setminus \Lambda)$ such that there exists $K > 0$, $q > 0$ such that for every $y \in \mathbb{R}^n \setminus \Lambda$,

$$K^{-1} \cdot d_0^{-q}(y, \Lambda) \leq f(y) \leq K \cdot d_0^{-q}(y, \Lambda), \quad (3.47)$$

where d_0 is the Euclidean distance on \mathbb{R}^n .

Lemma 3.12. Fix $0 < \alpha < 1$ and $\Lambda \subset \mathbb{R}^n$ be a closed subset of zero measure. Let $f \in L_{2\alpha}(\mathbb{R}^n) \cap C^{\infty}(\mathbb{R}^n \setminus \Lambda)$ and denote by f_{ϵ} the standard mollifier of f in the sense of (3.24). Then

$$\lim_{\epsilon \rightarrow 0} ((-\Delta)^{\alpha} f_{\epsilon})(x) = ((-\Delta)^{\alpha} f)(x). \quad (3.48)$$

for every $x \in \mathbb{R}^n \setminus \Lambda$.

Proof. First, by Definition 3.4,

$$(-\Delta)^{\alpha} f_{\epsilon}(x) = -C_{n,\alpha} \int_{\mathbb{R}^n} \frac{f_{\epsilon}(x+y) + f_{\epsilon}(x-y) - 2f_{\epsilon}(x)}{|y|^{n+2\alpha}} dy. \quad (3.49)$$

Let

$$H_{\epsilon}(x, y) \equiv \frac{f_{\epsilon}(x+y) + f_{\epsilon}(x-y) - 2f_{\epsilon}(x)}{|y|^{n+2\alpha}}, \quad (3.50)$$

and

$$H(x, y) \equiv \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{n+2\alpha}}. \quad (3.51)$$

Then for fixed $0 < \delta_0(x) < \frac{1}{10^3} \cdot d_0(x, \Lambda)$,

$$\begin{aligned} (-\Delta)^\alpha f_\epsilon(x) &= -C_{n,\alpha} \int_{B_{\delta_0}(0^n)} H_\epsilon(x, y) dy - C_{n,\alpha} \int_{\mathbb{R}^n \setminus B_{\delta_0}(0^n)} H_\epsilon(x, y) dy \\ &\equiv I_\epsilon^{(1)}(x) + I_\epsilon^{(2)}(x). \end{aligned} \quad (3.52)$$

Correspondingly,

$$I^{(1)}(x) \equiv -C_{n,\alpha} \int_{B_{\delta_0}(0^n)} \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{n+2\alpha}} dy, \quad (3.53)$$

and

$$I^{(2)}(x) \equiv -C_{n,\alpha} \int_{\mathbb{R}^n \setminus B_{\delta_0}(0^n)} \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{n+2\alpha}} dy. \quad (3.54)$$

So we will prove that for every $x \in \mathbb{R}^n \setminus \Lambda$,

$$\lim_{\epsilon \rightarrow 0} I_\epsilon^{(1)}(x) = I^{(1)}(x), \quad (3.55)$$

and

$$\lim_{\epsilon \rightarrow 0} I_\epsilon^{(2)}(x) = I^{(2)}(x). \quad (3.56)$$

The first stage is to prove (3.55). The proof follows from the derivatives convergence property in Lemma 3.8. In fact,

$$|I_\epsilon^{(1)}(x) - I^{(1)}(x)| = -C_{n,\alpha} \int_{B_{\delta_0}(0^n)} (H_\epsilon(x, y) - H(x, y)) dy, \quad (3.57)$$

and

$$\begin{aligned} |H_\epsilon(x, y) - H(x, y)| &= \frac{|(f_\epsilon - f)(x+y) + (f_\epsilon - f)(x-y) - 2(f_\epsilon - f)(x)|}{|y|^{n+2\alpha}} \\ &\leq C(n) \frac{\|\nabla^2(f_\epsilon - f)\|_{B_{2\delta_0}(x)}}{|y|^{n+2\alpha-2}} \\ &\leq C(n) \frac{\|\nabla^2 f\|_{B_{4\delta_0}(x)}}{|y|^{n+2\alpha-2}}. \end{aligned} \quad (3.58)$$

Notice that the function $|y|^{n+2\alpha-2} \in L^1(B_{\delta_0}(0^n))$. Hence, applying Lemma 3.7 and dominated convergence theorem, the convergence

$$\lim_{\epsilon \rightarrow 0} I_\epsilon^{(1)}(x) = I^{(1)}(x) \quad (3.59)$$

holds for every $x \in \mathbb{R}^n \setminus \Lambda$.

Next we switch to prove (3.56). The proof basically follows from Lemma 3.10. In fact,

$$\begin{aligned}
|I_\epsilon^{(2)}(x) - I^{(2)}(x)| &= C_{n,\alpha} \int_{\mathbb{R}^n \setminus B_{\delta_0}(0^n)} |H_\epsilon(x, y) - H(x, y)| dy \\
&\leq C_{n,\alpha} \left(\int_{\mathbb{R}^n \setminus B_{\delta_0}(0^n)} \frac{|(f_\epsilon - f)(x + y)|}{|y|^{n+2\alpha}} dy + \int_{\mathbb{R}^n \setminus B_{\delta_0}(0^n)} \frac{|(f_\epsilon - f)(x - y)|}{|y|^{n+2\alpha}} dy \right. \\
&\quad \left. + 2 \int_{\mathbb{R}^n \setminus B_{\delta_0}(0^n)} \frac{|f_\epsilon(x) - f(x)|}{|y|^{n+2\alpha}} dy \right) \\
&\equiv C_{n,\alpha} (D_\epsilon(x) + F_\epsilon(x) + 2G_\epsilon(x)).
\end{aligned} \tag{3.60}$$

Notice that, for every $x \in \mathbb{R}^n \setminus \Lambda$,

$$D_\epsilon(x) = F_\epsilon(x) = \int_{\mathbb{R}^n \setminus B_{\delta_0}(0^n)} \frac{|(f_\epsilon - f)(y)|}{|x - y|^{n+2\alpha}} dy, \tag{3.61}$$

and thus by Lemma 3.10,

$$\lim_{\epsilon \rightarrow 0} D_\epsilon(x) = \lim_{\epsilon \rightarrow 0} F_\epsilon(x) = 0. \tag{3.62}$$

On the other hand, by Lemma 3.7, for every $x \in \mathbb{R}^n \setminus \Lambda$,

$$\lim_{\epsilon \rightarrow 0} G_\epsilon(x) = 0. \tag{3.63}$$

Now the proof of (3.56) is done, and hence the proof of the lemma is complete. \square

Lemma 3.13. *Given $0 < \alpha < 1$ and $\tau > 0$, let f satisfy Assumption 3.11 and $f \in L_{loc}^\tau(\mathbb{R}^n)$. For every fixed $x \in \mathbb{R}^n \setminus \Lambda$, choose $\delta_0(x) \equiv \frac{1}{10^6} \cdot \min\{d_0(x, \Lambda), \delta_1\}$ such that $f(y) > 10^3$ for every $y \in T_{10\delta_1}(\Lambda) \equiv \{y \in \mathbb{R}^n | d_0(y, \Lambda) \leq 10\delta_1\}$. Then the following holds for every $x \in \mathbb{R}^n \setminus \Lambda$:*

(1)

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus (B_{\delta_0}(x) \cup T_{\delta_0}(\Lambda))} \frac{|(f_\epsilon)^\tau(y) - f^\tau(y)|}{|x - y|^{n+2\alpha}} dy = 0. \tag{3.64}$$

(2)

$$\lim_{\epsilon \rightarrow 0} \int_{B_{\delta_0}(0^n)} \frac{|((f_\epsilon)^\tau - f^\tau)(x + y) + ((f_\epsilon)^\tau - f^\tau)(x - y) - 2((f_\epsilon)^\tau - f^\tau)(x)|}{|y|^{n+2\alpha}} dy = 0. \tag{3.65}$$

(3)

$$\lim_{\epsilon \rightarrow 0} \int_{T_{\delta_0}(\Lambda)} \frac{|(f_\epsilon)^\tau(y) - f^\tau(y)|}{|x - y|^{n+2\alpha}} dy = 0. \tag{3.66}$$

(4)

$$\lim_{\epsilon \rightarrow 0} \left((-\Delta)^\alpha (f_\epsilon)^\tau \right)(x) = \left((-\Delta)^\alpha f^\tau \right)(x). \tag{3.67}$$

(5)

$$\lim_{\epsilon \rightarrow 0} \left((-\Delta)^\alpha \log(f_\epsilon) \right)(x) = \left((-\Delta)^\alpha \log f \right)(x). \tag{3.68}$$

Proof. (1) follows from C_0 -estimate of f . By Assumption 3.11, there exists $K_0 > 0$, $p > 0$ such that for every $y \in \mathbb{R}^n \setminus T_{\delta_0/2}(\Lambda)$, we have that

$$\frac{1}{K_0} \cdot (d_0(y, \Lambda))^{-\tau \cdot q} \leq f^\tau(y) \leq K_0 \cdot (d_0(y, \Lambda))^{-\tau \cdot q}, \quad (3.69)$$

and thus for every $y \in \mathbb{R}^n \setminus T_{\delta_0/2}(\Lambda)$

$$|f^\tau(y)| \leq 2^{\tau \cdot q} \cdot K_0 \cdot \delta_0^{-\tau \cdot q}. \quad (3.70)$$

By the definition of f_ϵ , it holds that for every $y \in \mathbb{R}^n \setminus T_{\delta_0}(\Lambda)$,

$$\begin{aligned} |(f_\epsilon)^\tau(y)| &= \left| \left(\int_{\mathbb{R}^n} f(y-z) \cdot \eta_\epsilon(z) dz \right)^\tau \right| \\ &= \left| \left(\int_{B_\epsilon(0^n)} f(y-z) \cdot \eta_\epsilon(z) dz \right)^\tau \right| \\ &\leq \|f\|_{L^\infty(B_\epsilon(y))} \cdot \left| \int_{B_\epsilon(0^n)} \eta_\epsilon(z) dz \right|^\tau \\ &= \|f\|_{L^\infty(B_\epsilon(y))}. \end{aligned} \quad (3.71)$$

Inequality (3.70) gives that for every $y \in \mathbb{R}^n \setminus T_{\delta_0}(\Lambda)$,

$$|(f_\epsilon)^\tau(y)| \leq 2^{\tau \cdot q} \cdot K_0 \cdot \delta_0^{-\tau \cdot q}. \quad (3.72)$$

Consequently,

$$\begin{aligned} \int_{\mathbb{R}^n \setminus (B_{\delta_0}(x) \cup T_{\delta_0}(\Lambda))} \frac{|(f_\epsilon)^\tau(y) - f^\tau(y)|}{|x-y|^{n+2\alpha}} dy &\leq \frac{2^{\tau \cdot q+1} \cdot K_0 \cdot \delta_0^{-\tau \cdot q}}{\delta_0^{\tau \cdot q}} \int_{\mathbb{R}^n \setminus B_{\delta_0}(x)} \frac{dy}{|x-y|^{n+2\alpha}} \\ &\leq \frac{2^{\tau \cdot q} \cdot K_0 \cdot \delta_0^{-\tau \cdot q - \alpha}}{\alpha}, \end{aligned} \quad (3.73)$$

and thus dominated convergence theorem implies that for every fixed $x \in \mathbb{R}^n \setminus \Lambda$,

$$\int_{\mathbb{R}^n \setminus (B_{\delta_0}(x) \cup T_{\delta_0}(\Lambda))} \frac{|(f_\epsilon)^\tau(y) - f^\tau(y)|}{|x-y|^{n+2\alpha}} dy \rightarrow 0. \quad (3.74)$$

(2) can be immediately proved by the C^2 -derivative estimate.

$$\begin{aligned} &\int_{B_{\delta_0}(0^n)} \frac{|((f_\epsilon)^\tau - f^\tau)(x+y) + ((f_\epsilon)^\tau - f^\tau)(x-y) - 2((f_\epsilon)^\tau - f^\tau)(x)|}{|y|^{n+2\alpha}} dy \\ &\leq \int_{B_{\delta_0}(0^n)} \frac{\|\nabla^2((f_\epsilon)^\tau - f^\tau)\|_{L^\infty(B_{2\delta_0}(x))}}{|y|^{n-2+2\alpha}} dy. \end{aligned} \quad (3.75)$$

By straightforward calculations,

$$\nabla^2((f_\epsilon)^\tau - f^\tau) = \tau \left((f_\epsilon)^{\tau-1} (\nabla^2 f_\epsilon) - f^{\tau-1} (\nabla^2 f) \right) + \tau(\tau-1) \left(f_\epsilon^{\tau-2} df_\epsilon \otimes df_\epsilon - f^{\tau-2} df \otimes df \right). \quad (3.76)$$

The pointwise convergence property of ∇f_ϵ and $\nabla^2 f_\epsilon$ implies that the integral (3.75) is finite. Therefore, dominated convergence theorem gives the desired convergence.

(3) follows from local integrability property, mean value theorem and Hölder's inequality. From now on, we assume that $\tau > 0$ and $\tau \neq 1$ because the case $\tau = 1$ has been proven in Lemma 3.12. Let

$$W(y) \equiv \tau \cdot \max\{|f_\epsilon|^{\tau-1}(y), |f|^{\tau-1}(y)\}, \quad (3.77)$$

and then by Lagrange mean value theorem, it holds that

$$\int_{T_{\delta_0}(\Lambda)} \frac{|(f_\epsilon)^\tau(y) - f^\tau(y)|}{|x - y|^{n+2\alpha}} dy \leq \int_{T_{\delta_0}(\Lambda)} \frac{W(y) \cdot |f_\epsilon - f|(y)}{|x - y|^{n+2\alpha}} dy. \quad (3.78)$$

Since $\delta_0(x) < \frac{d_0(x, \Lambda)}{10^6}$ and $d_0(y, \Lambda) < \delta_0$, so triangle inequality implies that

$$|x - y| = d_0(x, y) > \delta_0. \quad (3.79)$$

Therefore,

$$\int_{T_{\delta_0}(\Lambda)} \frac{|(f_\epsilon)^\tau(y) - f^\tau(y)|}{|x - y|^{n+2\alpha}} dy \leq \frac{1}{\delta_0^{n+2\alpha}} \int_{T_{\delta_0}(\Lambda)} W(y) \cdot |f_\epsilon - f|(y) dy. \quad (3.80)$$

(i) When $0 < \tau < 1$, by definition, $|W(y)| < 1$ for every $y \in T_{\delta_0}(\Lambda)$, which implies that

$$\int_{T_{\delta_0}(\Lambda)} \frac{|(f_\epsilon)^\tau(y) - f^\tau(y)|}{|x - y|^{n+2\alpha}} dy \leq \frac{1}{\delta_0^{n+2\alpha}} \int_{T_{\delta_0}(\Lambda)} |f_\epsilon - f|(y) dy \rightarrow 0. \quad (3.81)$$

(ii) Next, we consider the case $\tau > 1$. Let $p \equiv \tau$ and $q = \frac{\tau}{\tau-1}$, then $\frac{1}{p} + \frac{1}{q} = 1$. Now applying Hölder inequality to the right hand side of inequality (3.80), we have that

$$\int_{T_{\delta_0}(\Lambda)} \frac{|(f_\epsilon)^\tau(y) - f^\tau(y)|}{|x - y|^{n+2\alpha}} dy \leq \frac{1}{\delta_0^{n+2\alpha}} \left(\int_{T_{\delta_0}(\Lambda)} |W(y)|^q dy \right)^{1/q} \cdot \left(\int_{T_{\delta_0}(\Lambda)} |f_\epsilon - f|^p dy \right)^{1/p}. \quad (3.82)$$

Now we estimate the uniform bound (independent of ϵ) of the integral $\int_{T_{\delta_0}(\Lambda)} |W(y)|^q dy$. In fact,

$$\begin{aligned} \left(\int_{T_{\delta_0}(\Lambda)} |f_\epsilon|^{(\tau-1)q} dy \right)^{1/q} &= \left(\int_{T_{\delta_0}(\Lambda)} |f_\epsilon|^\tau dy \right)^{(\tau-1)/\tau} \\ &= \left(\int_{T_{\delta_0}(\Lambda)} |(f_\epsilon - f) + f|^\tau dy \right)^{(\tau-1)/\tau} \\ &\leq \left(\left(\int_{T_{\delta_0}(\Lambda)} |f_\epsilon - f|^\tau dy \right)^{1/\tau} + \left(\int_{T_{\delta_0}(\Lambda)} |f|^\tau dy \right)^{1/\tau} \right)^{\tau-1}. \end{aligned} \quad (3.83)$$

For $\epsilon \rightarrow 0$, it holds that

$$\int_{T_{\delta_0}(\Lambda)} |f_\epsilon - f|^\tau dy \rightarrow 0, \quad (3.84)$$

and then (3.83) implies that

$$\left(\int_{T_{\delta_0}(\Lambda)} |f_\epsilon|^{(\tau-1)q} dy \right)^{1/q} \leq 2^{\tau-1} \cdot \left(\int_{T_{\delta_0}(\Lambda)} |f|^\tau dy \right)^{(\tau-1)/\tau}. \quad (3.85)$$

Immediately by the definition of the function W , we have that

$$\left(\int_{T_{\delta_0}(\Lambda)} |W(y)|^{(\tau-1)q} dy \right)^{1/q} \leq 2^{\tau-1} \cdot \left(\int_{T_{\delta_0}(\Lambda)} |f(y)|^\tau dy \right)^{(\tau-1)/\tau}. \quad (3.86)$$

Therefore, when $\epsilon \rightarrow 0$, the above computations and (3.82) give that

$$\int_{T_{\delta_0}(\Lambda)} \frac{|(f_\epsilon)^\tau(y) - f^\tau(y)|}{|x - y|^{n+2\alpha}} dy \rightarrow 0. \quad (3.87)$$

The proof of (3) is done.

(4) easily follows from the combination of (1), (2) and (3).

Now we proceed to prove (5). First notice that, if f satisfies Assumption 3.11 and $f \in L_{loc}^\tau(\mathbb{R}^n)$ for some $\tau > 1$, then it holds that $\log f \in L_{2\alpha}(\mathbb{R}^n)$ and $\log(f_\epsilon) \in L_{2\alpha}(\mathbb{R}^n)$ which means that both $(-\Delta)^\alpha \log f$ and $(-\Delta)^\alpha \log(f_\epsilon)$ are well defined. In fact, for sufficiently small $\delta_0 > 0$, we have that $1 \ll \log f < f$ on $T_{\delta_0}(\Lambda)$ and thus $\log f \in L_{loc}^\tau(\mathbb{R}^n)$. In particular, $\log f \in L_{loc}^\tau(\mathbb{R}^n)$. Moreover, by Assumption 3.11,

$$K^{-1} \cdot d_0^{-q}(y, \Lambda) \leq f(y) \leq K \cdot d_0^{-q}(y, \Lambda), \quad (3.88)$$

and thus for sufficiently large $R \gg 1$, it holds that

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_R(0^n)} \frac{\log f(y)}{1 + |y|^{n+2\alpha}} dy &\leq \int_{\mathbb{R}^n \setminus B_R(0^n)} \frac{C \cdot \log |y|}{|y|^{n+2\alpha}} dy \\ &\leq \int_{\mathbb{R}^n \setminus B_R(0^n)} \frac{C|y|^m}{|y|^{n+2\alpha}} dy < \infty, \end{aligned} \quad (3.89)$$

where the exponent $m > 0$ can be chosen such that $0 < m < \alpha$. Therefore, $\log f \in L_{2\alpha}(\mathbb{R}^n)$. Similarly, one can show that $\log(f_\epsilon) \in L_{2\alpha}(\mathbb{R}^n)$.

With the above preliminary result, we will show the convergence property. The above arguments actually shows that for every $x \in \mathbb{R}^n \setminus \Lambda$,

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus (B_{\delta_0}(x) \cup T_{\delta_0}(\Lambda))} \frac{|\log(f_\epsilon)(y) - \log f(y)|}{|x - y|^{n+2\alpha}} dy = 0. \quad (3.90)$$

Moreover, the convergence of the integral in $B_{\delta_0}(0^n)$ is the same as that in (2). Now it suffices to prove that for every $x \in \mathbb{R}^n \setminus \Lambda$,

$$\lim_{\epsilon \rightarrow 0} \int_{T_{\delta_0}(\Lambda)} \frac{|(\log(f_\epsilon))(y) - \log f(y)|}{|x - y|^{n+2\alpha}} = 0. \quad (3.91)$$

The basic idea is the same as that in (3), for the completion we still give the detailed proof here. We choose $\delta_0(x) < \frac{1}{10^\epsilon} d_0(x, \Lambda)$ such that $f(y) > 10^3$ for every $y \in T_{10\delta_0}(\Lambda)$.

It is by definition that $f(y) > 0$ and $f_\epsilon(y) > 0$ for every $y \in \mathbb{R}^n$, so denote

$$W(y) \equiv \max\{(f_\epsilon)^{-1}(y), f^{-1}(y)\} \in (0, 1), \quad (3.92)$$

then Lagrangian mean value theorem and (3.92) imply that

$$\begin{aligned}
\int_{T_{\delta_0}(\Lambda)} \frac{|(\log(f_\epsilon))(y) - \log f(y)|}{|x - y|^{n+2\alpha}} &\leq \delta_0^{-n-2\alpha} \int_{T_{\delta_0}(\Lambda)} |W(y)| \cdot |f_\epsilon(y) - f(y)| dy \\
&\leq \delta_0^{-n-2\alpha} \int_{T_{\delta_0}(\Lambda)} |f_\epsilon(y) - f(y)| dy \\
&\stackrel{\epsilon \rightarrow 0}{\leq} 0.
\end{aligned} \tag{3.93}$$

The proof is done. \square

Example 3.1. Let (M^n, g) be a Kleinian manifold, then its universal cover $(\widetilde{M}^n, \tilde{g})$ satisfies $\tilde{g} = e^{2w} g_0$ where g_0 is the Euclidean metric. Denote by Λ the limit set of the Kleinian group $\Gamma \cong \pi_1(M^n)$. By lemma 2.4, the function

$$f(x) \equiv e^{w(x)}, \forall x \in \mathbb{R}^n \setminus \Lambda \tag{3.94}$$

satisfies Assumption 3.11.

Proposition 3.14. *Given $\tau_2 > \tau_1 > 0$, assume that $f \in L_{loc}^{\tau_2}(\mathbb{R}^n)$ satisfies Assumption 3.11, Then for every $x \in \mathbb{R}^n \setminus \Lambda$, we have*

$$((-\Delta)^\alpha f^{\tau_1})(x) \geq \frac{\tau_1}{\tau_2} \cdot f^{\tau_1-\tau_2}(x) \cdot \left((-\Delta)^\alpha f^{\tau_2} \right)(x). \tag{3.95}$$

Proof. Let f_ϵ be the standard mollifier of f for every $0 < \epsilon < 10^{-6}$. Then by Proposition 3.6,

$$((-\Delta)^\alpha (f_\epsilon)^{\tau_1})(x) \geq \frac{\tau_1}{\tau_2} \cdot (f_\epsilon)^{\tau_1-\tau_2}(x) \cdot \left((-\Delta)^\alpha (f_\epsilon)^{\tau_2} \right)(x). \tag{3.96}$$

Applying Lemma 3.12 and Lemma 3.13, we obtain the desired inequality by taking $\epsilon \rightarrow 0$. \square

4. HAUSDORFF DIMENSION BOUND OF THE LIMIT SET

In this section, we prove Theorem 1.1. The basic strategy is to bound the Hausdorff dimension of the limit set Λ by applying lemma 2.8. So the key is to prove the following integral estimate on the distance function to limit set Λ .

Proposition 4.1. *Given $n \geq 4$, let (M^n, g) be locally conformally flat such that $R_g > 0$ and $Q_{2\gamma} > 0$ for some $1 < \gamma < 2$. Denote by $\Lambda \equiv \Lambda(\Gamma)$ the limit set on \mathbb{S}^n . Let $0 < R < +\infty$ satisfy $\Lambda \subset B_R(0^n)$, then*

$$\int_{B_R(0^n) \setminus \Lambda} d_0^{-\frac{n+2\gamma}{2}}(x, \Lambda) < \infty. \tag{4.1}$$

Proof. Let $\tilde{g} = e^{2w}g_0$, then by the conformal covariance property of $P_{2\gamma}$,

$$P_{2\gamma}(1) = e^{-\frac{n+2\gamma}{2}w}(-\Delta)^\gamma(e^{\frac{n-2\gamma}{2}w}), \quad (4.2)$$

and thus by the definition of $Q_{2\gamma}$,

$$Q_{2\gamma} \equiv \left(\frac{n-2\gamma}{2}\right)^{-1} \cdot P_{2\gamma}(1) = \left(\frac{n-2\gamma}{2}\right)^{-1} e^{-\frac{n+2\gamma}{2}w}(-\Delta)^\gamma(e^{\frac{n-2\gamma}{2}w}). \quad (4.3)$$

Applying lemma 2.4 and equation (4.3), we have that

$$\begin{aligned} \int_{B_R(0^n) \setminus \Lambda} Q_{2\gamma} \cdot d_0^{-\frac{n+2\gamma}{2}}(x, \Lambda) &\leq C(n, \gamma, g) \int_{B_R(0^n) \setminus \Lambda} Q_{2\gamma} \cdot e^{\frac{n+2\gamma}{2}w} \\ &= C(n, \gamma, g) \int_{B_R(0^n) \setminus \Lambda} (-\Delta)^\gamma(e^{\frac{n-2\gamma}{2}w}). \end{aligned} \quad (4.4)$$

Denote by $\alpha \equiv \gamma - 1 \in (0, 1)$, $u \equiv (-\Delta)^\alpha(e^{\frac{n-2\gamma}{2}w})$ and define

$$U_\lambda \equiv \left\{ x \in B_R(0^n) \setminus \Lambda \mid u(x) > \lambda \right\}. \quad (4.5)$$

We will prove that the subsets U_λ exhausts $B_R(0^n) \setminus \Lambda$ when $\lambda \rightarrow +\infty$. That is,

$$B_R(0^n) \setminus \Lambda = \lim_{\lambda \rightarrow +\infty} U_\lambda. \quad (4.6)$$

It suffices to show that when $d_0(x, \Lambda) \rightarrow 0$,

$$u(x) \equiv (-\Delta)^\alpha(e^{\frac{n-2\gamma}{2}w})(x) \rightarrow +\infty. \quad (4.7)$$

In fact, by Proposition 3.14,

$$(-\Delta)^\alpha(e^{\frac{n-2\gamma}{2}w})(x) \geq \frac{n-2\gamma}{n-2\alpha} \cdot e^{(\alpha-\gamma)w(x)} \cdot \left((-\Delta)^\alpha e^{\frac{n-2\alpha}{2}w} \right)(x). \quad (4.8)$$

Applying the conformal covariance property of $P_{2\alpha}$, we have that

$$(-\Delta)^\alpha e^{\frac{n-2\alpha}{2}w} = \left(\frac{n-2\alpha}{2}\right) \cdot Q_{2\alpha} \cdot e^{\frac{n+2\alpha}{2}w}. \quad (4.9)$$

Plugging the above equation into inequality (4.8),

$$(-\Delta)^\alpha(e^{\frac{n-2\gamma}{2}w})(x) \geq \frac{n-2\gamma}{2} \cdot e^{\frac{n+4\alpha-2\gamma}{2}w(x)} \cdot Q_{2\alpha}(x). \quad (4.10)$$

Since we have chosen $\alpha \equiv \gamma - 1 \in (0, 1)$, immediately

$$\frac{n+4\alpha-2\gamma}{2} = \frac{n-2+2\alpha}{2} > 0. \quad (4.11)$$

Moreover, by theorem 1.2 in [GQ], there is some $c_0 > 0$ such that $Q_{2\alpha} \geq c_0 > 0$ provided M^n is compact and $R_g > 0$. Therefore,

$$(-\Delta)^\alpha(e^{\frac{n-2\gamma}{2}w})(x) \geq c_0 \cdot \frac{n-2\gamma}{2} \cdot e^{\frac{n-2+2\alpha}{2}w(x)}, \quad (4.12)$$

then lemma 2.4 implies that

$$\lim_{d_0(x, \Lambda) \rightarrow 0} u(x) = \lim_{d_0(x, \Lambda) \rightarrow 0} (-\Delta)^\alpha (e^{\frac{n-2\gamma}{2}w})(x) = +\infty. \quad (4.13)$$

In the above notations, it holds that $(-\Delta)^\gamma = (-\Delta) \circ (-\Delta)^\alpha$. Let us estimate the following integral over U_λ . Denote by $E_\lambda \equiv \{x \in B_R(0^n) \setminus \Lambda \mid u(x) = \lambda\}$ and immediately $E_\lambda \cup \partial B_R(0^n) = \partial U_\lambda$. Let ν be the unit normal vector field of ∂U_λ , then integration by parts gives that

$$\begin{aligned} \int_{U_\lambda} (-\Delta)^\gamma (e^{\frac{n-2\gamma}{2}w}) &= \int_{U_\lambda} -\Delta_0 u \\ &= - \int_{\partial U_\lambda} \langle \nabla_0 u, \nu \rangle \\ &= - \int_{E_\lambda} \langle \nabla_0 u, \nu \rangle - \int_{\partial B_R(0^n)} \langle \nabla_0 u, \nu \rangle. \end{aligned} \quad (4.14)$$

On the level set E_λ we have that $\nu = \frac{\nabla_0 u}{|\nabla_0 u|}$ (λ can be chosen as regular values), which implies that

$$\begin{aligned} \int_{U_\lambda} (-\Delta)^\gamma (e^{\frac{n-2\gamma}{2}w}) &= - \int_{E_\lambda} |\nabla_0 u| - \int_{\partial B_R(0^n)} \langle \nabla_0 u, \nu \rangle \\ &\leq C(n, g). \end{aligned} \quad (4.15)$$

Taking the limit $\lambda \rightarrow +\infty$, we have that

$$\int_{B_R(0^n) \setminus \Lambda} (-\Delta)^\gamma (e^{\frac{n-2\gamma}{2}w}) = \lim_{\lambda \rightarrow +\infty} \int_{U_\lambda} (-\Delta)^\gamma (e^{\frac{n-2\gamma}{2}w}) \leq C(n, g) < +\infty. \quad (4.16)$$

Plugging the above estimate and the assumption $Q_{2\gamma} \geq c_1 > 0$ into (4.4),

$$\int_{B_R(0^n) \setminus \Lambda} d_0^{-\frac{n+2\gamma}{2}}(x, \Lambda) \leq C(n, \gamma, g) < \infty. \quad (4.17)$$

So the proof of the proposition is complete. □

Proof of Theorem 1.1. First, by Proposition 4.1 and lemma 2.8, we have that

$$\dim_{\mathcal{H}}(\Lambda) \leq \frac{n-2\gamma}{2}. \quad (4.18)$$

Then standard continuity argument gives that there exists $\epsilon > 0$ such that for every $\gamma' \in (\gamma, \gamma + \epsilon)$ it holds that $Q_{2\gamma'} > 0$ which implies that

$$\dim_{\mathcal{H}}(\Lambda) \leq \frac{n-2\gamma'}{2} < \frac{n-2\gamma}{2}. \quad (4.19)$$

□

5. THE CRITICAL CASE: Q_3 CURVATURE ON 3-DIMENSIONAL MANIFOLDS

In this section, we will prove Theorem 1.3. The strategy of the proof is similar to that of Theorem 1.1. The new point is Proposition 5.1 which is a limiting version of Proposition 3.14.

Proposition 5.1. *Let $\Lambda \subset \mathbb{R}^3$ be a closed subset of zero measure and let e^w satisfy Assumption 3.11. Then for every $x \in \mathbb{R}^3 \setminus \Lambda$,*

$$((-\Delta)^{1/2}w)(x) \geq e^{-w(x)}((-\Delta)^{1/2}e^w)(x). \quad (5.1)$$

Proof. Denote by $f \equiv e^w$ and let f_ϵ the standard mollifier of f in the sense of (3.24). Let $1 < \gamma < 3/2$, by Proposition 3.6, then for every $x \in \mathbb{R}^3$

$$\frac{\tau_2}{\tau_1} \cdot \left((-\Delta)^{\gamma-1}(f_\epsilon)^{\tau_1} \right)(x) \geq (f_\epsilon)^{\tau_1-\tau_2}(x) \cdot \left((-\Delta)^{\gamma-1}(f_\epsilon)^{\tau_2} \right)(x), \quad (5.2)$$

where $\tau_1 = \frac{3-2\gamma}{2}$ and $\tau_2 = \frac{5-2\gamma}{2}$. At the first stage, we will prove that

$$\left((-\Delta)^{1/2}(\log(f_\epsilon)) \right)(x) \geq (f_\epsilon)^{-1}(x) \cdot (-\Delta)^{1/2}(f_\epsilon)(x). \quad (5.3)$$

To this end, let $\gamma \rightarrow \frac{3}{2}$ then $\tau_1 \rightarrow 0$ and $\tau_2 \rightarrow 1$, by (5.2), so it turns out that

$$\lim_{\gamma \rightarrow \frac{3}{2}} \frac{\tau_2}{\tau_1} \cdot \left((-\Delta)^{\gamma-1}(f_\epsilon)^{\tau_1} \right)(x) \geq (f_\epsilon)^{-1}(x) \cdot (-\Delta)^{1/2}(f_\epsilon)(x). \quad (5.4)$$

Notice that, by the continuity property of the fractional Laplacian operator, the left hand side satisfies the following

$$\begin{aligned} \lim_{\gamma \rightarrow \frac{3}{2}} \frac{\tau_2}{\tau_1} \cdot \left((-\Delta)^{\gamma-1}(f_\epsilon)^{\tau_1} \right)(x) &= \lim_{\gamma \rightarrow \frac{3}{2}} (-\Delta)^{\gamma-1} \left(\frac{\tau_2}{\tau_1} \cdot (f_\epsilon)^{\tau_1} \right)(x) \\ &= \lim_{\gamma \rightarrow \frac{3}{2}} (-\Delta)^{\gamma-1} \left(\frac{\tau_2}{\tau_1} \cdot ((f_\epsilon)^{\tau_1} - 1) \right)(x). \end{aligned} \quad (5.5)$$

It is straightforward that $(f_\epsilon)^{\tau_1} - 1 = \tau_1 \cdot \log(f_\epsilon) + \sum_{k=2}^{\infty} \frac{(\tau_1 \log(f_\epsilon))^k}{k!}$, which implies that

$$\lim_{\gamma \rightarrow 3/2} \frac{\tau_2}{\tau_1} \cdot ((f_\epsilon)^{\tau_1} - 1) = \log(f_\epsilon). \quad (5.6)$$

Consequently,

$$\lim_{\gamma \rightarrow \frac{3}{2}} \frac{\tau_2}{\tau_1} \cdot \left((-\Delta)^{\gamma-1}(f_\epsilon)^{\tau_1} \right)(x) = \left((-\Delta)^{1/2} \log(f_\epsilon) \right)(x). \quad (5.7)$$

Then (5.4) implies that,

$$\left((-\Delta)^{1/2} \log(f_\epsilon) \right)(x) \geq (f_\epsilon)^{-1}(x) \cdot (-\Delta)^{1/2}(f_\epsilon)(x). \quad (5.8)$$

So the proof of (5.3) is done.

Next, applying Lemma 3.12 and (5) in Lemma 3.13, we have that for every $x \in \mathbb{R}^3 \setminus \Lambda$,

$$((-\Delta)^{1/2}w)(x) \geq e^{-w(x)}((-\Delta)^{1/2}e^w)(x). \quad (5.9)$$

Now the proof of the proposition is complete. \square

As what we did in Section 4, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. By lemma 2.4 and lemma 2.8, it suffices to show that

$$\int_{B_R(0^3) \setminus \Lambda} e^{3w(x)} dx \leq C(g). \quad (5.10)$$

Let \hat{g} be the Riemannian metric on the universal cover of M^3 , and thus $\hat{g} = e^{2w} g_0$. Denote by $\hat{Q}_3 \equiv Q_{3,\hat{g}}$, then the conformal covariance property gives that,

$$\hat{Q}_3 \cdot e^{3w} = (-\Delta)^{3/2} w. \quad (5.11)$$

First, we will prove that for every $x \in \mathbb{R}^n \setminus \Lambda$,

$$\lim_{d(x,\Lambda) \rightarrow 0} ((-\Delta)^{1/2} w)(x) = +\infty. \quad (5.12)$$

In fact, by Proposition 5.1, for every $x \in \mathbb{R}^n \setminus \Lambda$,

$$((-\Delta)^{1/2} w)(x) \geq e^{-w(x)} ((-\Delta)^{1/2} e^w)(x). \quad (5.13)$$

Notice that conformal covariance property of P_1 implies that

$$(-\Delta)^{1/2} (e^w) = \hat{Q}_1 \cdot e^{2w}. \quad (5.14)$$

By Proposition 5.1,

$$((-\Delta)^{1/2} w)(x) \geq e^{-w(x)} \cdot (-\Delta)^{1/2} (e^w)(x) = e^{w(x)} \cdot Q_1(x). \quad (5.15)$$

Since $R_g > 0$, applying a result in [GQ], we have that $Q_1 > 0$ everywhere on M^3 . By assumption, M^3 is compact and so there is some $c_0 > 0$ such that

$$Q_1(x) \geq c_0 > 0, \quad \forall x \in M^3, \quad (5.16)$$

which implies that

$$\hat{Q}_1(x) \geq c_0 > 0, \quad \forall x \in \widetilde{M}^3, \quad (5.17)$$

Therefore,

$$((-\Delta)^{1/2} w)(x) \geq c_0 \cdot e^{w(x)}, \quad (5.18)$$

which implies that

$$\lim_{d_0(x,\Lambda) \rightarrow 0} ((-\Delta)^{1/2} w)(x) = +\infty. \quad (5.19)$$

The proof of (5.12) is done.

Now we are in a position to prove inequality (5.10). Let $u(x) \equiv ((-\Delta)^{1/2} w)(x)$ and

$$U_\lambda \equiv \{x \in B_R(0^n) \setminus \Lambda \mid u(x) \geq \lambda\}, \quad (5.20)$$

then

$$\begin{aligned}
\int_{U_\lambda} (-\Delta)^{3/2} w(x) dx &= - \int_{\partial U_\lambda} \langle \nabla_0 u, \nu_0 \rangle - \int_{\partial B_R(0^n)} \langle \nabla_0 u, \nu_0 \rangle \\
&= - \int_{\partial U_\lambda} |\nabla_0 u| - \int_{\partial B_R(0^n)} \langle \nabla_0 u, \nu_0 \rangle \\
&\leq C(g).
\end{aligned} \tag{5.21}$$

Since $\widehat{Q}_3(x) \geq c_0 > 0$, so we have that

$$\int_{B_R(0^n) \setminus \Lambda} e^{3w(x)} dx \leq \frac{1}{c_0} \int_{B_R(0^n) \setminus \Lambda} \widehat{Q}_3(x) \cdot e^{3w(x)} dx = \frac{1}{c_0} \lim_{\lambda \rightarrow +\infty} \int_{U_\lambda} (-\Delta)^{3/2} w(x) dx \leq C(g). \tag{5.22}$$

□

6. EXAMPLES

We will present in this section several examples with concrete Kleinian groups and we will compute the $Q_{2\gamma}$ curvature of them. Moreover, we will give an example which shows both the estimate in Theorem 1.1 and the assumption in Theorem 1.4 are in fact optimal.

Example 6.1 and Example 6.2 are rather standard spaces with positive $Q_{2\gamma}$ curvature. The detailed computations can be found in in [DGon].

Example 6.1 (Standard sphere). Given any $n \geq 3$, consider the sphere (\mathbb{S}^n, g_1) with $g_1 = (\frac{2}{1+|x|^2})^2 g_0$. Immediately, $\sec_{g_1} \equiv 1$ and $R_{g_1} \equiv n(n-1)$. Then for every $\gamma \in (0, n/2)$, by the conformal covariance property of $P_{2\gamma}$, we can compute that

$$P_{2\gamma}(1) \equiv \frac{\Gamma(\frac{n+2\gamma}{2})}{\Gamma(\frac{n-2\gamma}{2})} > 0. \tag{6.1}$$

It follows that for every $\gamma \in (0, n/2)$,

$$Q_{2\gamma} \equiv \left(\frac{n-2\gamma}{2}\right)^{-1} P_{2\gamma}(1) > 0. \tag{6.2}$$

Let $n = 3$ and $\gamma = \frac{n}{2} = \frac{3}{2}$,

$$Q_3 \equiv \lim_{\gamma \rightarrow 3/2} \left(\frac{3-2\gamma}{2}\right)^{-1} P_{2\gamma}(1) = \lim_{\gamma \rightarrow 3/2} \frac{\Gamma(\frac{3+2\gamma}{2})}{\frac{3-2\gamma}{2} \cdot \Gamma(\frac{3-2\gamma}{2})} = \Gamma(3) = 2. \tag{6.3}$$

Example 6.2 (Standard cylinder). Fix any $n \geq 3$, consider the compact manifold $(M^n, h) \equiv (\mathbb{S}^{n-1} \times \mathbb{S}^1, h)$ with the standard product metric such that $R_h \equiv (n-1)(n-2)$. To understand the behavior of $P_{2\gamma}$, we look at the universal cover $(\widetilde{M}^n, \widetilde{h}) \equiv (\mathbb{S}^{n-1} \times \mathbb{R}^1, \widetilde{h})$, where the covering metric \widetilde{h} can be written in terms of polar coordinates in \mathbb{R}^n , $\widetilde{h} \equiv \frac{g_{\mathbb{R}^n}}{r^2}$. So standard computations

imply that for every $\gamma \in (0, n/2)$,

$$P_{2\gamma}(1) = 2^{2\gamma} \frac{\Gamma^2(\frac{n+2\gamma}{4})}{\Gamma^2(\frac{n-2\gamma}{4})} > 0, \quad (6.4)$$

and hence by definition,

$$Q_{2\gamma} \equiv \left(\frac{n-2\gamma}{2}\right)^{-1} P_{2\gamma}(1) > 0, \quad \gamma \in (0, n/2). \quad (6.5)$$

Then it follows that in the critical case, namely, $n = 3$ and $\gamma = \frac{3}{2}$,

$$Q_3 \equiv \lim_{\gamma \rightarrow 3/2} \left(\frac{3-2\gamma}{2}\right)^{-1} P_{2\gamma}(1) = \lim_{\gamma \rightarrow 3/2} \frac{2^{2\gamma} \cdot \Gamma^2(\frac{3+2\gamma}{4})}{\frac{3-2\gamma}{2} \cdot \Gamma^2(\frac{3-2\gamma}{4})} = 0. \quad (6.6)$$

In the next example, the Kleinian group Γ has limit set $\Lambda(\Gamma)$ of positive Hausdorff dimension. From this example, one can see that the dimension estimate in Theorem 1.1 is sharp, and lower bound $\gamma = \frac{3}{2}$ in Theorem 1.4 is also optimal.

Example 6.3 (Limit set with positive dimension). Let $(M^5, \omega) \equiv (\mathbb{S}^3 \times \Sigma^2, g_1 \oplus g_{\Sigma^2})$ be a Riemannian product space, where (\mathbb{S}^3, g_1) is the 3-sphere with $\sec_{g_1} \equiv 1$ and (Σ^2, h_{Σ^2}) is a closed hyperbolic surface with $\sec_{h_{\Sigma^2}} \equiv -1$. Then by standard calculations, the scalar curvature is constant $R_\omega \equiv 4$ and (M^5, ω) is a locally conformally flat manifold. Indeed, the universal cover $(\widetilde{M}^5, \tilde{\omega}) \equiv (\mathbb{S}^3 \times \mathbb{H}^2, g_1 \oplus h_{-1})$ has $\sec_{g_1} \equiv 1$ and $\sec_{h_{-1}} \equiv -1$, which implies that

$$\tilde{\omega} = g_1 \oplus h_{-1} = g_1 + \frac{dy^2 + dx^2}{y^2} = \frac{g_{\mathbb{R}^5}}{y^2}, \quad y > 0. \quad (6.7)$$

Therefore, $(\widetilde{M}^5, \tilde{\omega})$ is conformally flat and $\widetilde{M}^5 \stackrel{\text{homeo}}{\cong} \mathbb{S}^5 \setminus \mathbb{S}^1$. It is a standard fact from hyperbolic geometry that $\Gamma \equiv \pi_1(M^5) = \pi_1(\Sigma^2)$ is actually of Fuchsian type such that

$$\dim_{\mathcal{H}}(\Lambda(\Gamma)) = 1 = \frac{5-3}{2}, \quad (6.8)$$

and $M^5 \stackrel{\text{homeo}}{\cong} (\mathbb{S}^5 \setminus \mathbb{S}^1)/\Gamma$.

Next, the universal covering space $(\widetilde{M}^5, \tilde{\omega})$ can be explicitly written as the conformal infinity of the 6-dimensional hyperbolic space $(\mathbb{H}^6, g_{\mathbb{H}^6})$ with $\sec_{g_{\mathbb{H}^6}} \equiv -1$. In fact, $g_{\mathbb{H}^6}$ can be represented as follows,

$$g_{wp} = ds^2 + \sinh^2(s)g_1 + \cosh^2(s)g_{-1}, \quad (6.9)$$

where g_1 is the metric on \mathbb{S}^3 with $\sec_{g_1} \equiv 1$ and g_{-1} is the metric on \mathbb{H}^2 with $\sec_{g_{-1}} \equiv -1$. Under the coordinate change $s = -\log(\frac{r}{2})$, then we have that

$$g_{wp} = \frac{dr^2 + (1 - \frac{r^2}{4})^2 g_1 + (1 + \frac{r^2}{4})^2 g_{-1}}{r^2}. \quad (6.10)$$

Indeed, with the above geodesic defining function r , $(\widetilde{M}^5, \tilde{\omega})$ can be viewed as the conformal infinity of (\mathbb{H}^6, g_{-1}) .

Now we are in a position to understand the fractional order curvatures of (M^5, ω) . By the standard computations in Fourier transform (see [GonMaSi] for details), for the above conformal factor y in (6.7), one can see that for $0 < \gamma < \frac{3}{2}$,

$$(-\Delta)^\gamma y^{-\frac{5-2\gamma}{2}} = c_0(\gamma) \cdot y^{-\frac{5+2\gamma}{2}}, \quad (6.11)$$

where

$$c_0(\gamma) \equiv \frac{\pi^{2\gamma-4}}{16} \cdot \frac{\Gamma(\frac{3+2\gamma}{4}) \cdot \Gamma(\frac{5+2\gamma}{4})}{\Gamma(\frac{5-2\gamma}{4}) \cdot \Gamma(\frac{3-2\gamma}{4})}. \quad (6.12)$$

Since $\tilde{\omega} = \frac{g_R^5}{y^2}$, by the conformal covariance property of $P_{2\gamma} \equiv P_{2\gamma}[\tilde{\omega}]$, we have that

$$P_{2\gamma}(1) = y^{\frac{5+2\gamma}{2}} (-\Delta)^\gamma y^{-\frac{5-2\gamma}{2}} = c_0(\gamma), \quad \forall 0 < \gamma < \frac{3}{2}. \quad (6.13)$$

Hence by definition,

$$Q_{2\gamma} \equiv \frac{2c_0(\gamma)}{5-2\gamma} > 0, \quad \forall 0 < \gamma < \frac{3}{2}, \quad (6.14)$$

which implies that

$$Q_3 = \lim_{\gamma \rightarrow 3/2} Q_{2\gamma} = \lim_{\gamma \rightarrow 3/2} c_0(\gamma) = 0. \quad (6.15)$$

Generally, by Theorem 1.1, for every $\frac{3}{2} \leq \gamma < 2$, M^5 does not admit any conformally flat metric with $Q_{2\gamma} > 0$ and $R > 0$. Moreover, Chang-Hang-Yang's work in [CHY] shows that M^5 does not admit any locally conformally flat metric with $R_h > 0$ and $Q_4 > 0$.

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